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THE ECONOMICS OF URBAN YARD SPACE: AN 'IMPLICIT-MARKET' MODEL FOR HOUSING ATTRIBUTES

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Summary

This paper develops the first theoretical housing model embodying the approach of the recent hedonic price literature. Instead of focusing on a scalar "housing service" consumption measure, the model portrays housing as a commodity with two attributes: floor space and yard space. Developers react to an hedonic price function, which relates dwelling rent to floor space, yard space, and location, in choosing the profit-maximizing characteristics of their housing complexes. The spatial behavior of the developer's choice variables is investigated (an interesting question is whether yards are larger farther from the urban center), and a comparative static analysis of the housing market equilibrium is presented.

The Economics of Urban Yard Space: An 'Implicit-Market' Model for Housing Attributes

by

Jan K. Brueckner*

1. Introduction

The last decade has witnessed a peculiar divergence between theoretical and empirical work in housing economics. A hallmark of the thriving theoretical literature on residential location begun by Alonso [1], Muth [7], and Mills [6] is the assumption that consumers purchase a unidimensional commodity called "housing services," which is produced with inputs of land and capital. In contrast, the burgeoning empirical hedonic price literature (see, for example, Kain and Quigley [5] and Grether and Mieszkowski [4]) is based on the assumption that housing is a multifaceted commodity whose consumption cannot be captured by a scalar housing service measure. Although Rosen [8] recently provided an abstract analysis of the market for such a commodity, which he viewed as providing an "implicit" market for the commodity's many attributes, a detailed theoretical <u>housing</u> model embodying the implicit market approach has not been developed. The purpose of the present paper is to tie together the divergent strands of the housing economics literature by providing such a framework. An urban spatial model is developed in which consumers value two housing attributes: floor space and yard space. Housing developers take account of the resulting hedonic price function, which relates dwelling rent to floor space, yard space, and location, in choosing the profit-maximizing characteristics of their housing complexes. A principal goal of the analysis is to describe the spatial behavior of the developer's choice variables. For example, an important question is whether yard sizes in the model realistically increase with distance to the central business district (CBD). The paper also presents a detailed comparative static analysis of the housing market equilibrium, establishing that many of the results derived by Wheaton [9] for a simple urban economy where land and a composite commodity are the only consumption goods emerge also in a more complicated and realistic model.

By making yard space a choice variable of the housing developer, the analysis in this paper also eliminates an unrealistic feature of the standard residential location model. In the standard model, housing services (which are best thought of as square feet of floor space) are produced with inputs of capital N and land ℓ according to a constant returns function $H(N,\ell)$. Now in order for this production function to be well-defined, the fraction of the land input physically covered by the structure must be specified in advance. This follows because a fixed amount of capital which completely covers a given land area will generate more square feet of floor space than the same amount of capital arranged vertically in a taller structure which occupies only a fraction of the available land (some capital will be used up in stairways, stronger foundations, etc.). To insure that a unique output of floor space is associated with each input bundle, buildings must occupy some constant fraction, say 75% or 100%, of the land area rented by the developer. In the latter case, buildings are yardless, and while the model will accurately represent the downtown areas of most large cities, it will fail to capture an essential feature of suburban land-use. Indeed, since the proportion of the land input covered by structures appears to decrease with distance to the CBD in American cities, any model in which this proportion is fixed will be at odds with reality. By allowing yard size to be a choice variable of the developer, the model developed below yields a potentially more accurate representation of an urban economy.

Section 2 of the paper presents the basic model and derives the optimality conditions. Section 3 analyses the spatial behavior of choice variables, and section 4 presents the market equilibrium comparative statics. Section 5 briefly considers the possibility of a corner solution to the optimization problem, and section 6 contains a summary and conclusions.

2. The Model

The first assumption in the analysis is that the (identical) urban consumers possess the strictly quasi-concave utility function $v(m,q,c)$, where m is consumption of a numeraire non-housing commodity, q is consumption of housing floor space, and c is consumption of yard space. Since all urban residents receive the same income y at their CBD workplace, the urban equilibrium must yield a uniform utility level for all residents. That is, all urban consumption bundles must satisfy $v(m,q,c) = u$, where u is the uniform utility level. Since $v_1 > 0$, this relationship may be inverted to yield $m = z(q,c,u)$, where $z_1 = -v_2/v_1 < 0$, $z_2 = -v_3/v_1 < 0$, and $z_3 = 1/v_1 > 0$. The function $z(q,c,u)$ indicates the amount of the non-housing good required to yield a utility level u for an individual with given consumption levels of floor and yard space. Now the rental payment for a given q-c bundle must leave the consumer an amount of income sufficient to purchase just enough of the numeraire good to reach utility level u. The rent R for a dwelling which provides q and c worth of floor and yard space and is located x miles from the CBD must therefore satisfy

$$y - t(x) - R = z(q,c,u), \qquad (1)$$

where $t(x)$ is commuting cost from distance x. Dwelling rent as a function of q and c and the vector $\theta \equiv (x,u,y)$ is consequently

$$R(q,c,\theta) \equiv y - t(x) - z(q,c,u). \qquad (2)$$

Eq. (2) is, of course, a hedonic price function which, for given income and utility levels, relates dwelling rent to floor space, yard space, and location. The fact that the utility function is strictly quasi-concave implies that z is a strictly convex function of q and c and hence that R is a strictly concave function of q and c. While this is easily seen from a diagram, an analytical proof is straightforward.[2]

Housing developers will select the characteristics of their output to maximize profit, taking account of the hedonic price function (2). The amount of floor space in a developer's complex is given by $H(N,\ell_1)$, where N is the capital input, ℓ_1 is the amount of land <u>physically covered by structures</u> (referred to subsequently as "building land"), and H is strictly concave and homogeneous of degree one. Since q equals floor space per dwelling, it follows that the number of dwellings in the complex equals $H(N,\ell_1)/q$. Now the consumption of yard space by each resident will depend on the total amount of land ℓ_2 devoted to yard space in the complex. It is assumed that yard space is like a private good in that consumption per resident is equal to ℓ_2 divided by the number of residents in the complex: $c = \ell_2 q/H(N,\ell_1)$. While it might be appropriate to assume that yard space is a pure public good ($c = \ell_2$) or that it constitutes an intermediate case between the extremes of pure public and private goods ($c = \ell_2[H(N,\ell_1)/q]^{-\gamma}$, $0 < \gamma < 1$),[3] it will be shown below that the developer's optimization problem has no solution in either of these cases.[4]

Given the preceeding discussion, the developer's profit equals

$$\frac{H(N,\ell_1)}{q} R(q,c,\theta) - r(\ell_1 + c\frac{H(N,\ell_1)}{q}) - nN, \tag{3}$$

where n is the (exogenous) unit rental price of capital and r is the (endogenous) land rent per acre. Note that HR/q is total revenue for the complex and that $r(\ell_1 + cH/q)$ equals total land cost. Recalling that H exhibits constant returns, (3) may be written more compactly as

$$\ell_1[\frac{h(S)}{q}(R(q,c,\theta) - cr) - nS - r], \tag{4}$$

where $S \equiv N/\ell_1$ is structural density and $h(S) = H(S,1)$, with $h'(S) = H_1(S,1) > 0$ and $h''(S) = H_{11}(S,1) < 0$. Note that the quantity in brackets in (4), denoted π, is profit per acre of building land.

For any given ℓ_1, developers choose S, c, and q to maximize (4), and competition bids up land rent r until maximized profit equals zero. Developers are then indifferent to the value of ℓ_1; the size of housing complexes is indeterminate. The first-order conditions for choice of structural density, yard space per dwelling, and floor space per dwelling are respectively

$$\frac{\partial\pi}{\partial S} \equiv \frac{h'(S)}{q}(R(q,c,\theta) - cr) - n = 0 \tag{5}$$

$$\frac{\partial\pi}{\partial c} \equiv \frac{h(S)}{q}(R_2(q,c,\theta) - r) = 0 \tag{6}$$

$$\frac{\partial\pi}{\partial q} \equiv \frac{h(S)}{q}[R_1(q,c,\theta) - \frac{(R(q,c,\theta) - cr)}{q}] = 0, \tag{7}$$

and the zero profit condition is

$$\pi \equiv \frac{h(S)}{q}(R(q,c,\theta) - cr) - nS - r = 0. \qquad (8)$$

Eq. (5) says that structural density is expanded until the marginal increase in revenue per acre of building land ($h'R/q$) equals the marginal increase in cost from the extra capital (n) plus the marginal increase in yard land cost required to hold yard space per dwelling fixed ($h'cr/q$). Eq. (6) says that yard space per dwelling is expanded until the marginal increase in dwelling rent (R_2) equals the marginal increase in yard land cost per dwelling (r). Eq. (7) says that dwelling size is expanded until the marginal decrease in revenue per acre of building land ($-hR_1/q + hR/q^2$) equals the marginal decrease in yard land cost from holding yard space per dwelling fixed (hcr/q^2).

Using (6) and (7) to eliminate terms, the Hessian matrix of π evaluated at the solution to (5)-(7) may be written

$$\begin{bmatrix} h''(R-cr)/q & 0 & 0 \\ 0 & hR_{22}/q & hR_{21}/q \\ 0 & hR_{21}/q & hR_{11}/q \end{bmatrix} \equiv A \qquad (9)$$

It is easy to see that the negative definiteness of A required by the second-order condition is guaranteed by $h'' < 0$ and the strict concavity of R ($R_{11}, R_{22} < 0$ and $R_{11}R_{22} - R_{21}^2 > 0$).

Having characterized the solution to the developer's optimization problem, it is interesting to note that the problem has no solution when yard space affords more jointness-in-consumption than a private good. Letting $c = \ell_2[H(N,\ell_1)/q]^{-\gamma}$, $0 \leq \gamma < 1$, (4) becomes

$$\ell_1[\frac{h(S)}{q}(R(q,c,\theta) - cr(\frac{h(S)}{q}\ell_1)^{\gamma-1}) - nS - r]. \qquad (10)$$

Since profit per acre of building land in (10) is increasing in ℓ_1, it follows that (10) is increasing in ℓ_1 for all possible values of the variables S, c, q, and r, indicating that the developer's optimization problem does not have a solution. Excessive jointness-in-consumption creates increasing returns to scale, leading to a familiar non-existence result. The intuition behind this result is especially clear in the pure public good case. Suppose that for an arbitrary value of ℓ_1, maximized profit equals zero. By increasing ℓ_1, holding S, c, and q fixed, the developer can increase profit per acre of building land, driving total profit above zero. This is possible because perfect jointness-in-consumption ($c = \ell_2$) means that total yard land may be held constant without reducing c as the population of the complex grows, with the result that as ℓ_1 increases, a fixed yard land cost may be spread over a larger number of acres of building land, increasing profit per acre. In the private good case, profit per acre of building land is independent of ℓ_1, which means that once profit is driven to zero, the developer has nothing to gain by increasing the size of his complex.

3. The Spatial Behavior of S, c, and q

It is well known that in the standard urban model, structural density and dwelling floor space are respectively decreasing and increasing functions of distance from the CBD, undeniably realistic results (see Muth [7]). In this section, the analysis focuses on the spatial behavior of S, c, and q in the present model.

The first-order conditions (5)-(7) and the zero-profit condition (8) together yield solutions for the variables S, c, q, and r in terms of the parameters n and $\theta \equiv [x,u,y]$. To compute comparative static derivatives with respect to x for S, c, and q, the first step is to determine the effect of x on r using (8). Differentiating (8) with respect to θ recalling $\frac{\partial\pi}{\partial S} = \frac{\partial\pi}{\partial c} = \frac{\partial\pi}{\partial q} = 0$ yields

$$\frac{\partial\pi}{\partial\theta} + \frac{\partial\pi}{\partial r}\frac{\partial r}{\partial\theta} = 0 \tag{11}$$

or, after substitution and rearrangement,

$$\frac{\partial r}{\partial\theta} = \frac{1}{\omega}\frac{\partial R}{\partial\theta}, \tag{12}$$

where $\omega \equiv c + q/h$ is yard land plus building land per dwelling (recall that h/q equals dwellings per acre of building land). Since $\partial R/\partial x = -t'(x)$ from (2), (12) yields $\partial r/\partial x = -t'/\omega < 0$. Totally differentiating (5)-(7) taking account of the dependence of r on x and solving for $\partial S/\partial x$, $\partial c/\partial x$, and $\partial q/\partial x$ then gives the following results:[6]

$$\frac{\partial S}{\partial x} = \frac{-h'\partial r/\partial x}{h|A|}\frac{h^2}{q^2}(R_{11}R_{22} - R_{21}^2) = \frac{h't'q}{hh''\omega(R-cr)} \tag{13}$$

$$\frac{\partial c}{\partial x} = \frac{h''(R-cr)\partial r/\partial x}{q|A|}\frac{h^2}{q^2}(R_{11} - R_{21}/h) = \frac{t'}{\omega}\frac{R_{21}/h - R_{11}}{R_{11}R_{22} - R_{21}^2} \tag{14}$$

$$\frac{\partial q}{\partial x} = \frac{t'}{\omega}\frac{R_{21} - R_{22}/h}{R_{11}R_{22} - R_{21}^2} \tag{15}$$

The implications of (13)-(15) may be stated as follows:

<u>Theorem 1</u>: While $\partial S/\partial x < 0$ holds for all x, the only constraint on the spatial behavior of floor and yard space is that the inequalities $\partial c/\partial x \leq 0$ and $\partial q/\partial x \leq 0$ cannot both be satisfied.

The first part of the theorem, which follows directly from (13) given $h'' < 0$ and $R-cr > 0$ (see (8)), means that structural density is a decreasing function of x; as in the standard model, buildings have fewer storeys farther from the CBD. The second part of the theorem says that c and q cannot both be (locally) nonincreasing functions of x; if yard space is locally nonincreasing in x, then floor space must be locally increasing in x, and vice versa. Of course, both c and q may increase locally in x without violating the theorem. The second part of the theorem is proved by noting that since R is concave, $\partial c/\partial x \leq 0$ and $\partial q/\partial x \leq 0$ require $R_{21}/h - R_{11} \leq 0$ and $R_{21} - R_{22}/h \leq 0$ respectively (see (14) and (15)). Recalling $R_{11}, R_{22} < 0$, it is clear that satisfaction of both inequalities requires $R_{21} < 0$, which means that the first inequality may be rewritten as $hR_{11}/R_{21} \leq 1$. Then, noting that the expression on the LHS is positive, it follows given $R_{22} < 0$ that

$$\frac{hR_{11}}{R_{21}} \frac{R_{22}}{h} \geq \frac{R_{22}}{h} \geq R_{21}, \tag{16}$$

where the latter inequality follows from the second original inequality. But, recalling $R_{21} < 0$, (16) reduces to $R_{11}R_{22} - R_{21}^2 \leq 0$, contradicting the strict concavity of R. Therefore, $\partial c/\partial x \leq 0$ and $\partial q/\partial x \leq 0$ cannot both be satisfied.

Since urban economists are used to thinking of variables which are monotonic functions of x, it is useful to state the implications of

Theorem 1 when c and q are monotonic functions. The theorem says that three possibilities are admissible: yard space per dwelling decreases with x while floor space per dwelling increases with x; yard space increases while floor space decreases with x; both yard space and floor space increase with x. While no data on yard sizes in urban areas are readily available, the latter possibility, which states that houses and yards are bigger farther from the CBD, appears to be confirmed by casual observation.

Although some restriction is placed on urban spatial structure by Theorem 1, the actual spatial behavior of floor and yard space under the model depends in part on the magnitude of R_{21}. While $R_{21} \geq 0$ is sufficient to yield $\partial c/\partial x$, $\partial q/\partial x > 0$ (see (14) and (15)), the sign of R_{21} is unfortunately ambiguous in general and is indeed negative under plausible restrictions on the utility function. To see this, note that

$$R_{21} \equiv \frac{-\partial z_1}{\partial c} \equiv \frac{\partial}{\partial c}\frac{v_2(z,q,c)}{v_1(z,q,c)}$$

$$= \frac{1}{v_1^3}\left(v_{11}v_2v_3 + v_{23}v_1^2 - v_{13}v_1v_2 - v_{12}v_1v_3\right). \qquad (17)$$

While (17) obviously cannot be signed in general, if the utility function is $\tau(m) + \phi(q,c)$, with $\tau'' < 0$ and $\phi_{12} < 0$, then (17) implies $R_{21} < 0$, yielding ambiguous signs for $\partial c/\partial x$ and $\partial q/\partial x$ (see (14) and (15)). Note that $\phi_{12} < 0$ is a natural assumption since it states that the incremental utility from extra floor space is less the larger the consumption of yard space. It is easy to show that $R_{21} < 0$ also holds in the less restrictive case where the utility function is $\delta(m,\phi(q,c))$, with $\phi_{12} < 0$ and δ quasi-concave in m and ϕ. Finally, it is worth noting that imposing the very strong restriction that the utility function is additively separable in all its arguments with a constant marginal utility of m ($v_{11} = 0$) yields $R_{21} \equiv 0$ and $\partial c/\partial x$, $\partial q/\partial x > 0$ (see (17)).

Given the ambiguity surrounding the general spatial behavior of floor and yard space, it is worthwhile to examine a special case by solving the first-order conditions for particular utility and production functions. Under the assumptions that $v(m,q,c) \equiv m^{\alpha}q^{\sigma}c^{\varepsilon}$ and $H(N,\ell_1) \equiv N^{\beta}\ell_1^{1-\beta}$, which gives $h(S) \equiv S^{\beta}$, considerable manipulation yields the following solutions for c and q:

$$c = \Omega(y-t(x))^{-\frac{\alpha+\beta\sigma}{(1-\beta)\sigma+\varepsilon}}$$
$$q = \Lambda(y-t(x))^{\frac{\beta\varepsilon-(1-\beta)\alpha}{(1-\beta)\sigma+\varepsilon}}, \tag{18}$$

where Ω and Λ are constants. Since all parameters are positive and $\beta < 1$, (18) implies that $\partial c/\partial x > 0$ and that

$$\frac{\partial q}{\partial x} \gtrless 0 \text{ as } \beta \lessgtr \frac{\alpha}{\alpha+\varepsilon}\,. \tag{19}$$

Under the Cobb-Douglas assumptions, yard space per dwelling is always increasing in x, while floor space per dwelling may be increasing, constant, or decreasing in x depending on the relationship between production and utility function parameters.

A final observation is that the ambiguous spatial behavior of floor and yard space in the model applies also to population density; the derivative $\partial\omega^{-1}/\partial x \equiv \partial(c + q/h)^{-1}/\partial x$ cannot be signed in general (recall that ω is total land area per dwelling). Under the Cobb-Douglas assumptions, however, population density realistically decreases with distance to the

CBD (see Muth [7] for empirical evidence). The population density solution is

$$\omega^{-1} = \Gamma(y-t(x))^{\frac{\alpha+\beta\sigma}{(1-\beta)\sigma+\varepsilon}}, \tag{20}$$

a decreasing function of x (Γ is a constant).

4. Market Equilibrium Comparative Statics

In this section of the paper, comparative static analysis of the housing market equilibrium is presented. The emergence of results similar to those derived by Wheaton [9] for the simplest type of urban economy (where land and a composite commodity are the only goods consumed) suggests the important conclusion that the comparative static properties of urban models are essentially unrelated to their level of detail and complexity.

Although the urban utility level enters parametrically in the developer's optimization problem, u is endogenously determined in a closed city, in which population P is fixed.[5] Recalling that r, S, c, and q (and hence ω) are functions of n, y, and u as a result of (5)-(8) and denoting agricultural land rent by r_A, the familiar equilibrium conditions which solve for u and $\bar{x}$, the distance to the urban boundary, as functions of the exogenous parameters $[P,r_A,y,n] \equiv \lambda$ are

$$\int_0^{\bar{x}} (2\pi x/\omega)dx = P \tag{21}$$

$$\bar{r} = r_A \tag{22}$$

Eq. (21) says that the urban population just fits inside the urban boundary, while (22) says that urban and agricultural land rents are equal at the boundary (the upper bar on r indicates that the variable is evaluated at $\bar{x}$).

The goal of the following analysis is to compute $\partial u/\partial\lambda$ and $\partial\bar{x}/\partial\lambda$. This task is greatly simplified by noting that the earlier result $\partial r/\partial x = -t'/\omega$ allows (21) to be rewritten as $-\int_0^{\bar{x}} (2\pi x/t')\frac{\partial r}{\partial x}\, dx = P$. Integrating by parts and substituting (22) allows (21) to be rewritten again as

$$\frac{\bar{x}r_A}{\bar{t}'} - \int_0^{\bar{x}} r\psi dx = -\frac{P}{2\pi}, \qquad (23)$$

where $\psi(x) \equiv \frac{1}{t'(x)}\left(1 - \frac{xt''(x)}{t'(x)}\right)$. The subsequent analysis requires $\psi > 0$, which may be guaranteed by the natural assumption $t'' \leq 0$. Together with $t(0) \geq 0$, this assumption realistically implies that commuting cost per mile is a decreasing function of trip length. The last step is to differentiate (23) with respect to λ and cancel terms involving $\partial\bar{x}/\partial\lambda$, which gives

$$\int_0^{\bar{x}} \left(\frac{\partial r}{\partial\lambda} + \frac{\partial r}{\partial u}\frac{\partial u}{\partial\lambda}\right)\psi dx = \frac{\bar{x}}{\bar{t}'}\frac{\partial r_A}{\partial\lambda} + \frac{1}{2\pi}\frac{\partial P}{\partial\lambda}, \qquad (24)$$

or, noting that $\partial u/\partial\lambda$ is independent of x,

$$\frac{\partial u}{\partial\lambda} = \frac{\frac{\bar{x}}{\bar{t}'}\frac{\partial r_A}{\partial\lambda} + \frac{1}{2\pi}\frac{\partial P}{\partial\lambda} - \int_0^{\bar{x}}\frac{\partial r}{\partial\lambda}\psi dx}{\int_0^{\bar{x}}\frac{\partial r}{\partial u}\psi dx} \qquad (25)$$

(this derivation follows Wheaton [9]). Now, from (12) and (2), $\partial r/\partial u = R_4/\omega = -z_3/\omega < 0$. Furthermore, computing $\partial r/\partial y$ from (12) and making a similar calculation to compute $\partial r/\partial n$ yields

$$\frac{\partial r}{\partial \lambda} = \frac{1}{\omega}[0 \quad 0 \quad 1 \quad -Sq/h]. \tag{26}$$

Then, noting $\partial P/\partial\lambda = [1\ 0\ 0\ 0]$ and $\partial r_A/\partial\lambda = [0\ 1\ 0\ 0]$, the following set of results emerges from simple inspection of (25):

<u>Theorem 2</u>: If $\psi > 0$, then

$$\frac{\partial u}{\partial P} < 0,\ \frac{\partial u}{\partial r_A} < 0,\ \frac{\partial u}{\partial y} > 0, \text{ and } \frac{\partial u}{\partial n} < 0.$$

The urban utility level decreases when the urban population or the cost parameters r_A and n increase, and increases when the urban income level increases. These intuitively sensible conclusions are identical to those derived by Wheaton.

To compute $\partial\bar{x}/\partial\lambda$, both sides of (22) are differentiated, yielding

$$\frac{\partial\bar{r}}{\partial x}\frac{\partial\bar{x}}{\partial\lambda} + \frac{\partial\bar{r}}{\partial\lambda} + \frac{\partial\bar{r}}{\partial u}\frac{\partial u}{\partial\lambda} = \frac{\partial r_A}{\partial\lambda} \tag{27}$$

or

$$\frac{\partial\bar{x}}{\partial\lambda} = \left(\frac{\partial r_A}{\partial\lambda} - \frac{\partial\bar{r}}{\partial u}\frac{\partial u}{\partial\lambda} - \frac{\partial\bar{r}}{\partial\lambda}\right)/\frac{\partial\bar{r}}{\partial x} \tag{28}$$

In order to evaluate $\partial\bar{x}/\partial\lambda$, the following results are needed:

<u>Lemma</u>: If c is a normal good, then $-R_{11}R_{24} + R_{21}R_{14} > 0$, and if q is a normal good, then $-R_{22}R_{14} + R_{12}R_{24} > 0$. Hence, if both c and q are normal goods,

$$\frac{dR_4}{dx} \equiv \frac{t'}{\omega(R_{11}R_{22}-R_{21}^2)}[-R_{11}R_{24} + R_{21}R_{14} + (-R_{22}R_{14} + R_{21}R_{24})/h] > 0.$$

To establish these results, the comparative static derivatives $\partial c/\partial y$ and $\partial q/\partial y$ from the problem max $v(m,q,c)$ s.t. $m + p_q q + p_c c = y$ are computed and shown to have the signs of $-R_{11}R_{24} + R_{21}R_{14}$ and $-R_{22}R_{14} + R_{12}R_{24}$ respectively. $\frac{dR_4}{dx} \equiv R_{41}\frac{\partial q}{\partial x} + R_{42}\frac{\partial c}{\partial x}$ is computed using (14) and (15).

Using the lemma, it is possible to establish the following conclusions regarding $\partial \bar{x}/\partial \lambda$:

<u>Theorem 3</u>: Suppose $\psi > 0$. Then $\partial \bar{x}/\partial P > 0$. Suppose further that c and q are normal goods. Then $\partial \bar{x}/\partial y > 0$. Suppose in addition that $\partial \omega/\partial x > 0$. Then $\partial \bar{x}/\partial r_A < 0$.

The theorem says that the city increases in area when population increases provided only that $\psi > 0$. If in addition c and q are normal goods, then the area of the city is an increasing function of income. The additional assumption $\partial \omega/\partial x > 0$ (total land area per dwelling increases with x) implies that the area of the city is a decreasing function of r_A. The first part of the theorem is established by noting from (28) that

$$\frac{\partial \bar{x}}{\partial P} = \frac{-\partial \bar{r}}{\partial u}\frac{\partial u}{\partial P} \Big/ \frac{\partial \bar{r}}{\partial x} > 0, \tag{29}$$

where the inequality follows from Theorem 2 and earlier results on the partial derivatives of r. To sign $\partial \bar{x}/\partial y$, $\partial u/\partial y$ from (25) is used in (28) to give

$$\frac{\partial \bar{r}}{\partial x}\frac{\partial \bar{x}}{\partial y} = \frac{\frac{\partial \bar{r}}{\partial u}\int_0^{\bar{x}} \frac{\partial r}{\partial y}\psi dx}{\int_0^{\bar{x}} \frac{\partial r}{\partial u}\psi dx} - \frac{\partial \bar{r}}{\partial y}, \tag{30}$$

which, recalling $\partial r/\partial u < 0$, has the sign of

$$\int_0^{\bar{x}} \left(\frac{\partial\bar{r}}{\partial y}\frac{\partial r}{\partial u} - \frac{\partial r}{\partial y}\frac{\partial\bar{r}}{\partial u}\right)\psi dx =$$

$$\int_0^{\bar{x}} \left(\frac{1}{\bar{\omega}}\frac{R_4}{\omega} - \frac{1}{\omega}\frac{\bar{R}_4}{\bar{\omega}}\right)\psi dx = \qquad (31)$$

$$\int_0^{\bar{x}} \frac{1}{\omega\bar{\omega}}(R_4 - \bar{R}_4)\psi dx.$$

When c and q are normal goods, the lemma gives $dR_4/dx > 0$, implying that $R_4 \leq \bar{R}_4$ holds over the range of integration in (31). Therefore (31) and hence (30) is negative, and $\partial\bar{x}/\partial y > 0$ follows from $\partial\bar{r}/\partial x < 0$.

Similarly, using (28) and (25), $\frac{\partial\bar{r}}{\partial x}\frac{\partial\bar{x}}{\partial r_A}$ equals

$$1 - \frac{\frac{\partial\bar{r}}{\partial u}\frac{\bar{x}}{\bar{t}'}}{\int_0^{\bar{x}} \frac{\partial r}{\partial u}\psi dx}\ , \qquad (32)$$

which has the sign of

$$\frac{\partial\bar{r}}{\partial u}\frac{\bar{x}}{\bar{t}'} - \int_0^{\bar{x}} \frac{\partial r}{\partial u}\psi dx\ . \qquad (33)$$

Integrating the last term by parts recalling $\frac{d}{dx}\frac{x}{t'} = \psi$ allows (33) to be rewritten as

$$\int_0^{\bar{x}} \frac{d}{dx}\left(\frac{\partial r}{\partial u}\right)\frac{x}{t'}\,dx = \int_0^{\bar{x}} \frac{d(R_4/\omega)}{dx}\frac{x}{t'}\,dx\ . \qquad (34)$$

Since $R_4 < 0$, it is easily seen that $dR_4/dx > 0$ and $\partial\omega/\partial x > 0$ together imply that the integrand in (34) is positive, making (32) positive and yielding $\partial\bar{x}/\partial r_A < 0$.

It is interesting to note that since $\partial\bar{x}/\partial n$ has the sign of

$$\int_0^{\bar{x}} \frac{1}{\omega\bar{\omega}} (R_4\overline{Sq}/\bar{h} - \bar{R}_4 Sq/h)\psi dx, \qquad (35)$$

$\partial\bar{x}/\partial n < 0$ will follow if $\partial(Sq/h)/\partial x > 0$. However, since it may be shown that $\partial(S/h)/\partial x < 0$, no natural restriction on $\partial q/\partial x$ will yield a determinate sign for $\partial\bar{x}/\partial n$.

It is also possible to describe how the variables S, c, and q are affected by changes in P, r_A, and y:

<u>Theorem 4</u>: Given $\psi > 0$,

(A) i) dS/dP, $dS/dr_A > 0$.

ii) When c is a normal good, $\partial c/\partial P < 0$ and $\partial c/\partial r_A < 0$ hold wherever $\partial c/\partial x \geq 0$.

iii) When q is a normal good, $\partial q/\partial P < 0$ and $\partial q/\partial r_A < 0$ hold wherever $\partial q/\partial x \geq 0$.

(B) When c and q are normal goods:

i) $dS^0/dy < 0$, $d\bar{S}/dy > 0$.

ii) At least one of the inequalities $\partial c^0/\partial y > 0$ and $\partial q^0/\partial y > 0$ must hold.

Part A of the theorem says that structural density increases at all locations when the urban population or agricultural land rent increases. Furthermore, with normality, c and q decrease with P and r_A at any location where these variables are increasing in x. For example, if c and q are both monotonically increasing functions of x, then c and q will decrease at all locations when the urban population or agricultural land rent increases. Part A of the theorem is proved by performing calculations similar to those used to derive $\partial S/\partial x$, $\partial c/\partial x$, and $\partial q/\partial x$ above to compute

$$\frac{\partial S}{\partial u} = \frac{-h'qR_4}{hh''\omega(R-cr)} < 0 \tag{36}$$

$$\frac{\partial c}{\partial u} = \frac{-R_{24}R_{11} + R_{21}R_{14} + (R_4/\omega)(R_{11}-R_{21}/h)}{R_{11}R_{22} - R_{21}^2} \tag{37}$$

$$\frac{\partial q}{\partial u} = \frac{-R_{22}R_{14} + R_{21}R_{24} + (R_4/\omega)(R_{22}/h-R_{21})}{R_{11}R_{22} - R_{21}^2} \tag{38}$$

Noting (36) and recalling $\partial u/\partial P < 0$, $dS/dP = (\partial S/\partial u)(\partial u/\partial P) > 0$ follows immediately. Similarly, $\partial u/\partial r_A < 0$ gives $dS/dr_A > 0$. Recalling the earlier lemma and noting (14) and (15), it follows from (37) and (38) that when c and q are normal goods, the inequalities $\partial c/\partial u > 0$ and $\partial q/\partial u > 0$ are satisfied wherever $\partial c/\partial x$, $\partial q/\partial x > 0$. Therefore $dc/dP = (\partial c/\partial u)(\partial u/\partial P)$ and $dq/dP = (\partial q/\partial u)(\partial u/\partial P)$ are negative wherever $\partial c/\partial x$, $\partial q/\partial x > 0$, with a parallel result holding with r_A in place of P. Note that since $\partial c/\partial x \leq 0$ and $\partial q/\partial x \leq 0$ cannot hold simultaneously by Theorem 1, Theorem 4 implies that at least one of the inequalities $dc/dP < 0$ and $dq/dP < 0$ must be satisfied at each location, with an analogous result for r_A.

Part B of Theorem 4 says that when c and q are normal goods and income increases, structural density falls at the city center (S^0 is S evaluated at x = 0) and rises at the old urban boundary, indicating that at some intermediate location, $dS/dy = 0$. Moreover, when y increases, c or q or both must increase at the city center. Part B is proved by first computing $\partial\overline{S}/\partial y$ and using (36) and (25) to evaluate

$$\frac{d\overline{S}}{dy} = \frac{\partial\overline{S}}{\partial y} + \frac{\partial\overline{S}}{\partial u}\frac{\partial u}{\partial y} = -\frac{\overline{h'}\overline{q}}{\overline{hh''}\overline{\omega}(\overline{R}-\overline{cr})}\left(1 + \overline{R}_4\frac{\partial u}{\partial y}\right). \tag{39}$$

Now (39) has the sign of

$$1 + \overline{R}_4 \frac{\partial u}{\partial y} = \frac{\int_0^{\bar{x}} \frac{1}{\omega}(R_4 - \overline{R}_4)\psi dx}{\int_0^{\bar{x}} \frac{1}{\omega} R_4 \psi dx} > 0, \tag{40}$$

recalling $dR_4/dx > 0$ and $R_4 < 0$. Repeating the argument with R_4^0 in place of $\overline{R}_4$ establishes $dS^0/dy < 0$.

Similar calculations yield

$$\frac{dc^0}{dy} = B_c + \left(\frac{R_{22}/h - R_{21}}{\omega(R_{11}R_{22} - R_{21}^2)}\right)^0 \left(1 + R_4^0 \frac{\partial u}{\partial y}\right) \tag{41}$$

where B_c is a positive quantity. Replacing B_c by $B_q > 0$ in (41) and $R_{22}/h - R_{21}$ by $R_{11} - R_{21}/h$ gives dq^0/dy. Now from above, $1 + R_4^0 \frac{\partial u}{\partial y} < 0$, and since at least one of $(R_{22}/h - R_{21})^0 < 0$ and $(R_{11} - R_{21}/h)^0 < 0$ must hold (see Theorem 1), it follows that at least one of $dc^0/dy > 0$ and $dq^0/dy > 0$ must hold.[6]

5. A Corner Solution for Yard Space

While it is an obvious fact that large portions of American central cities are essentially yardless, the analysis so far does not admit this possibility since only interior solutions to the developer's problem ($c > 0$) have been considered. This section briefly considers the possibility of a corner solution, where yard space per dwelling equals zero. To handle this case, eq. (6) among the original first-order conditions must be replaced by the three Kuhn-Tucker conditions

$$c \geq 0$$

$$(R_2 - r) \leq 0 \qquad (42)$$

$$c(R_2 - r) = 0.$$

First, it is useful to derive the spatial behavior of S and q over a range of x where c = 0. Using (5), (7), and (8) with c set equal to zero, it is easily shown that $\partial S/\partial x < 0$ and $\partial q/\partial x > 0$ hold at all locations where c = 0, results which are familiar from the standard model. Note that if the central part of the city is realistically yardless, then while S will decrease with x at all locations, yardless or otherwise (recall Theorem 1), and q will increase with x in the yardless central part of the city, the spatial behavior of q at more distant locations will be constrained only by Theorem 1, with $\partial q/\partial x < 0$ admissible.

To investigate the issue of whether yardless housing will indeed be centrally located, suppose that the necessary conditions (5), (7), (8), and (42) hold at some x' with c = 0. If it can be shown that the necessary conditions are satisfied with c = 0 for all x < x', then it will follow that the solution requires c = 0 for x < x' and hence that the range of x values where c = 0 (if one exists) will be of the form $[0,\hat{x}]$. This result will emerge if $\frac{d}{dx}(R_2 - r)\Big|_{c=0} > 0$, since $(R_2 - r)\Big|_{c=0} \leq 0$ at x' will then yield $(R_2 - r)\Big|_{c=0} < 0$ for x < x', implying that if the necessary conditions hold with c = 0 at x = x', they hold with c = 0 for all x < x'. Unfortunately, the above inequality need not be satisfied, making the location of the yardless part of the city ambiguous in general. It may be shown, however, that if

$(R_{11} - R_{21}/h)\Big|_{c=0} < 0$, then $\frac{d}{dx}(R_2 - r)\Big|_{c=0} > 0$, guaranteeing that if the city has a yardless area, it will be centrally located.

6. Summary and Conclusion

This paper has constructed the first theoretical housing model incorporating the hedonic approach common to many recent empirical studies. In the model, consumers value two housing attributes, floor space and yard space, and developers react to the resulting hedonic price function in choosing profit-maximizing characteristics for their housing complexes. Since the model is explicitly spatial, the spatial behavior of the developer's choice variables is of special interest. While it was shown that structural density in the model is a decreasing function of distance to the CBD, a result familiar from the standard approach, the spatial behavior of floor space and yard space per dwelling was shown to be subject only to the rather weak restriction that both variables cannot decrease with distance to the CBD. This result means that while dwelling sizes might decrease with distance, contrary to the predictions of the standard model, such a pattern must always be accompanied by larger yards farther from the CBD.

Comparative static analysis of the urban equilibrium showed that results similar to those demonstrated by Wheaton [9] for the simplest type of urban economy may be derived in a much more detailed and realistic model. In particular, it was shown that the urban utility level in a closed city is a decreasing function of population, agricultural land rent, and capital cost and an increasing function of income. In addition, the land area occupied by the city was shown under reasonable assumptions to be an increasing function of population and income and a decreasing function of agricultural rent.

Finally, it was noted that for the model to apply to typically yardless American central cities, the housing developer must be viewed as achieving a corner solution with zero yard space per dwelling in such areas. A potential criticism of the model centers on this point. It might be argued that urban land-use patterns are a result of competition between developers producing two distinct commodities, single- and multi-family housing, according to different technologies. Multi-family housing, which, it might be argued, is typically yardless as a result of consumer aversion to shared yard space, could be located centrally as a result of the shape of its bid-rent curve, with single-family housing found at more distant locations. While this explanation of land-use patterns is not without appeal, its defect consists of a failure to treat housing as a single commodity whose characteristics at a given location reflect the response of the housing developer to locational attributes such as accessibility to the CBD. The present model, which embodies such an approach, offers a more unified view of urban spatial structure.

Footnotes

*I wish to thank Jon Sonstelie for helpful comments. Any errors, however, are my own.

[1]Arnott and MacKinnon [2] briefly consider a model somewhat similar to the one developed below.

[2]Since v is strictly quasi-concave $\alpha v(m_0,q_0,c_0) + (1-\alpha)v(m_1,q_1,c_1) < v(m^*,q^*,c^*)$, where $m^* = \alpha m_0 + (1-\alpha)m_1$ and similarly for q^* and c^*, and where $v(m_0,q_0,c_0) = v(m_1,q_1,c_1) = u$. Now since $v(z(q^*,c^*,u),q^*,c^*) \equiv u$ and $v_1 > 0$, it follows from above that $m^* > z(q^*,c^*,u)$. But since $m_i = z(q_i,c_i,u)$, $i=0,1$, it follows from the definitions of m^*, q^*, and c^* that $\alpha z(q_0,c_0,u) + (1-\alpha)z(q_1,c_1,u) > z(\alpha q_0 + (1-\alpha)q_1, \alpha c_0 + (1-\alpha)c_1,u)$, establishing that z is a strictly convex function of q and c.

[3]Recent work in local public finance recognizes that most public goods exhibit congestion: holding output Q fixed, individual consumption b decreases with the size T of the consuming group. A convenient functional representation of this phenomenon is $b = QT^{-\gamma}$; $\gamma = 0$ corresponds to a pure public good while $\gamma = 1$ means the good is private, with output divided equally among members of the consuming group. For $0 < \gamma < 1$, congestability lies between the extremes of pure public and private goods. For a fuller treatment of public good congestion, see Brueckner [3].

[4]Another possibility which is not considered is that for fixed ℓ_2, yard consumption per household may first increase and then decrease with the number of yard users. Increased safety from crime or greater potential for recreational activities might explain the range of increasing consumption.

[5]To save space, analysis for an open city, where utility is exogenous and population endogenous, is not presented.

[6]The ambiguity of the signs of $\frac{\partial S}{\partial n}$, $\frac{\partial c}{\partial n}$, and $\frac{\partial q}{\partial n}$ made it impossible to deduce the total effects of an increase in n on S, c, and q.

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